Radiated power by relativistically accelerating charge (Liénard's formula, generalisation of the Larmor formula):

$$\boldsymbol{P}_{\text{rad}} = \frac{2k_e q^2}{3c^3}\gamma^6\left(a^2 - \left(\frac{\vec{v}\times\vec{a}}{c}\right)^2\right) = \frac{2k_e q^2}{3c^3}\gamma^6\left(a^2 - \frac{v^2 a^2 \sin^2\theta_{va}}{c^2}\right) = \frac{2k_e q^2 a^2}{3c^3}\frac{1-\frac{v^2}{c^2}\sin^2\theta_{va}}{\left(1-\frac{v^2}{c^2}\right)^3}$$

$$= \frac{2k_e q^2 a^2}{3c^3}\frac{1-\beta^2\sin^2\theta_{va}}{(1-\beta^2)^3}$$

Special cases:

Linear acceleration ($\vec{v} \parallel \vec{a}$):

$$\theta_{va} = 0 \qquad \therefore \sin^2\theta_{va} = 0 \qquad \boldsymbol{P}_{\parallel} = \frac{2k_e q^2 a^2}{3c^3}\frac{1}{(1-\beta^2)^3} = \frac{2k_e q^2 a^2}{3c^3}\gamma^6$$

Circular (perpendicular) acceleration ($\vec{v} \perp \vec{a}$):

$$\theta_{va} = 90° \qquad \therefore \sin^2\theta_{va} = 1 \qquad \boldsymbol{P}_{\perp} = \frac{2k_e q^2 a^2}{3c^3}\frac{1}{(1-\beta^2)^2} = \frac{2k_e q^2 a^2}{3c^3}\gamma^4$$

Didn't Einstein say free fall is inert motion? If so, will a charge that's freely falling towards another (one of opposite sign) then radiate? If not, we can ignore the above. However, doesn't Einstein's *Equivalence Principle* say *gravitation* is fully equivalent to *acceleration*? Then it *should* radiate and this *should* slow down the acceleration, right? Shouldn't the latter then influence the free fall? It's called the Abraham-Lorentz-Dirac *force*.

We'll restrict ourselves to a *charged mass* with: (m, q)
in radial free fall towards another one with: $(M \ggg m, Q)$

using the *parallel-accelerated radiated power*: $P_{\parallel} = \frac{2k_e q^2 a^2}{3c^3}\frac{1}{(1-\beta^2)^3} = \frac{2k_e q^2 a^2}{3c^3}\gamma^6$

We've got: $a = \frac{dv}{dt} = \dot{\beta}c$

yielding: $\boldsymbol{P}_{\text{rad}}(\beta,\dot{\beta}) = \frac{2k_e q^2 a^2}{3c^3}\frac{1}{(1-\beta^2)^3} = \frac{2k_e q^2}{3c}\frac{\dot{\beta}^2}{(1-\beta^2)^3}$

We also have: $\beta^2 = \frac{\gamma^2-1}{\gamma^2} \therefore \beta = \pm\frac{\sqrt{\gamma^2-1}}{\gamma} \therefore \dot{\beta} = \frac{\dot{\gamma}}{\gamma^2\sqrt{\gamma^2-1}} \therefore \dot{\beta}^2 = \frac{\dot{\gamma}^2}{\gamma^4(\gamma^2-1)}$

hence: $\boldsymbol{P}_{\text{rad}}(\gamma,\dot{\gamma}) = \frac{2k_e q^2\dot{\beta}^2}{3c}\gamma^6 = \frac{2k_e q^2}{3c}\frac{\dot{\gamma}^2}{\gamma^4(\gamma^2-1)}\gamma^6 = \frac{2k_e q^2}{3c}\frac{\gamma^2\dot{\gamma}^2}{\gamma^2-1}$

We convert *power* to *force* by dividing it by the *velocity* $\boldsymbol{v} = \boldsymbol{\beta c}$:

so: $\left|\boldsymbol{F}_{\text{rad}}(\beta,\dot{\beta})\right| = \left|\frac{2k_e q^2}{3c^2}\frac{\dot{\beta}^2}{\beta(1-\beta^2)^3}\right|$

if: $\beta < 0$ (falling inwards)

then: $F_{\text{rad}}>0$ (*braking force* outwards)

hence: $\boldsymbol{F}_{\text{rad}}(\beta,\dot{\beta}) = -\frac{2k_e q^2}{3c^2}\frac{\dot{\beta}^2}{\beta(1-\beta^2)^3}$ (pos. if $\beta < 0$, i.e. falling inwards)

Also: $\left|\boldsymbol{F}_{\text{rad}}(\gamma,\dot{\gamma})\right| = \left|\frac{2k_e q^2}{3c^2}\frac{1}{\beta}\frac{\gamma^2\dot{\gamma}^2}{\gamma^2-1}\right| = \frac{2k_e q^2}{3c^2}\frac{\gamma^3\dot{\gamma}^2}{\sqrt{\gamma^2-1}(\gamma^2-1)}$

i.e.: $\boldsymbol{F}_{\text{rad}}(\gamma,\dot{\gamma}) = \frac{2k_e q^2}{3c^2}\frac{\gamma^3\dot{\gamma}^2}{\sqrt{\gamma^2-1}(\gamma^2-1)}$ (always pos.)

Gravitation[1]: $\boldsymbol{F}_{\text{gr}} = \frac{-GMmc^4}{(rc^2-GM)(rc^2-Gm)}$ (*Newsteinian*)

[1] See: https://henk-reints.nl/astro/HR-Newstein-corrected.pdf eqn.39 @p.7 (as of 2026-09-03)

and *electric force*[2]: $\boldsymbol{F}_{\mathrm{el}} = \frac{-k_e Qq}{(r-r_Q)(r-r_q)} = \frac{-k_e Qq}{\left(r-\frac{\sqrt{Gk_e}}{c^2}Q\right)\left(r-\frac{\sqrt{Gk_e}}{c^2}q\right)} = \frac{-k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})}$

yield a ***total force*** of: $\boldsymbol{F}_{\mathrm{tot}} = \boldsymbol{F}_{\mathrm{gr}} + \boldsymbol{F}_{\mathrm{el}} + \boldsymbol{F}_{\mathrm{rad}}$

so: $\boldsymbol{F}_{\mathrm{tot}} = \frac{-GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{-k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} - \frac{2k_e q^2}{3c^2}\frac{\dot\beta^2}{\beta(1-\beta^2)^3}$

or: $\boldsymbol{F}_{\mathrm{tot}} = \frac{-GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{-k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} + \frac{2k_e q^2}{3c^2}\frac{\gamma^3\dot\gamma^2}{\sqrt{\gamma^2-1}(\gamma^2-1)}$

as a function of: $M, Q, m, q, r, \beta, \dot\beta$

or: $M, Q, m, q, r, \gamma, \dot\gamma$

We'll further use the first of these equations for $\boldsymbol{F}_{\mathrm{tot}}$.

Target is to find r as a function of t.

We assumed: $M \ggg m$

so we can use: $F = \dot p = \frac{d(\gamma m v)}{dt} = \frac{d(\gamma m\beta c)}{dt} = mc\frac{d(\gamma\beta)}{dt} = mc(\dot\gamma\beta + \gamma\dot\beta)$

We have: $\gamma = (1-\beta^2)^{-1/2} =: W^{-1/2}$

Chain rule: $\frac{d\gamma}{dt} = \frac{d\gamma}{dW}\frac{dW}{d\beta}\frac{d\beta}{dt}$

renders: $\dot\gamma = \frac{-W^{-3/2}}{2}(-2\beta)\dot\beta = \frac{-(1-\beta^2)^{-3/2}}{2}(-2\beta)\dot\beta = \frac{\beta\dot\beta}{(1-\beta^2)^{3/2}}$

& we obtain: $F = mc\left(\frac{\beta\dot\beta}{(1-\beta^2)^{3/2}}\beta + (1-\beta^2)^{-1/2}\dot\beta\right)$

which is: $F = mc\left(\frac{\beta^2\dot\beta}{(1-\beta^2)^{3/2}} + \frac{\dot\beta}{(1-\beta^2)^{1/2}}\right) = mc\dot\beta\left(\frac{\beta^2}{(1-\beta^2)^{3/2}} + \frac{1}{(1-\beta^2)^{1/2}}\right)$

hence: $\boldsymbol{F} = mc\dot\beta\left(\frac{\beta^2}{(1-\beta^2)^{3/2}} + \frac{1-\beta^2}{(1-\beta^2)^{3/2}}\right) = \frac{mc\dot\beta}{(1-\beta^2)^{3/2}} = mc\gamma^3\dot\beta$

The rightmost is Einstein's first eqn. @p.919 of *Zur Elektrodynamik bewegter Körper* (he [rather confusingly to us] used symbol β instead of γ).

This means: $\boldsymbol{F}_{\mathrm{tot}} = \frac{mc\dot\beta}{(1-\beta^2)^{3/2}} = \frac{-GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{-k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} - \frac{2k_e q^2}{3c^2}\frac{\dot\beta^2}{\beta(1-\beta^2)^3}$

i.e.: $\frac{mc\dot\beta}{(1-\beta^2)^{3/2}} + \frac{2k_e q^2}{3c^2}\frac{\dot\beta^2}{\beta(1-\beta^2)^3} + \frac{GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} = 0$

With: $\beta = \frac{v}{c} = \frac{\dot r}{c}$

we get: $\frac{mc\dot\beta}{\left(1-\left(\frac{\dot r}{c}\right)^2\right)^{3/2}} + \frac{2k_e q^2}{3c^2}\frac{\dot\beta^2}{\left(\frac{\dot r}{c}\right)\left(1-\left(\frac{\dot r}{c}\right)^2\right)^3} + \frac{GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} = 0$

Shouldn't it be that: $\dot\beta = \frac{d}{dt}\left(\frac{v}{c}\right) = \frac{1}{c}\frac{dv}{dt} = \frac{1}{c}\frac{d\dot r}{dt} = \frac{\ddot r}{c}$

as perceived by a distant stationary & inert observer**?**

Then: $\frac{m\ddot r}{\left(\frac{c^2-\dot r^2}{c^2}\right)^{3/2}} + \frac{2k_e q^2}{3c^3}\frac{\ddot r^2}{\dot r\left(\frac{c^2-\dot r^2}{c^2}\right)^3} + \frac{GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} = 0$

i.e.: $\frac{mc^3\ddot r}{(c^2-\dot r^2)^{3/2}} + \frac{2k_e q^2 c^3\ddot r^2}{3\dot r(c^2-\dot r^2)^3} + \frac{GMmc^4}{(rc^2-GM)(rc^2-Gm)} + \frac{k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})} = 0$

should be the ***equation of motion*** of a *light* ***charged mass***
attracted by some *heavy* ***charged mass***
under the condition than $M \ggg m$ & $\dot r < 0$.

[2] See: https://henk-reints.nl/astro/HR-Graviticity.pdf

It describes a *Newsteinian* radial free fall ("bull's eye trajectory")
as perceived by a *distanert* (distant, stationary & inert) observer.
Since $\boldsymbol{F}_{\mathbf{el}}$ already incorporates the correct sign for an attractive force,
$\boldsymbol{Q}$ & $\boldsymbol{q}$ must be the *absolute values* of the two opposite (i.e. attractive!) charges.
Finding a general analytical solution seems rather hard to me
(my mathematical skills have severly shrunk during the last few decades),
but it should not be too hard to numerically solve the thing.

The D.E. is: $$\frac{2k_e q^2 c^3 \ddot{r}^2}{3\dot{r}(c^2-\dot{r}^2)^3}+\frac{mc^3\ddot{r}}{(c^2-\dot{r}^2)^{3/2}}+\frac{GMmc^4}{(rc^2-GM)(rc^2-Gm)}+\frac{k_e Qqc^4}{(rc^2-Q\sqrt{Gk_e})(rc^2-q\sqrt{Gk_e})}=0$$

Possible algorithm:

input: $\boldsymbol{M},\ \boldsymbol{m},\ \boldsymbol{Q},\ \boldsymbol{q},\ \boldsymbol{r_0},\ \dot{\boldsymbol{r}}_0,\ \Delta t$;
write all input values to output file (with full precision to enable exact reproduction!);
if ($\mathrm{isNaN}(\boldsymbol{M})$): $\boldsymbol{M}=\sqrt{\boldsymbol{k_e Q^2/G}}$; //# use the *charge mass* (see ref./footnote 2)
if ($\mathrm{isNaN}(\boldsymbol{m})$): $\boldsymbol{m}=\sqrt{\boldsymbol{k_e q^2/G}}$; //# sic
//# in Newtonian mechanics, *total* & *reduced mass* can be used;
//# ~~D.V.~~ N.V. that's accurate enough here as well...
//# if ($\boldsymbol{M}\ggg\boldsymbol{m}$) it'll certainly suffice, since the D.E. was built thereupon:
$\boldsymbol{\mathcal{M}}=\boldsymbol{M}+\boldsymbol{m}$; //# *total mass*
$\boldsymbol{\mu}=\boldsymbol{Mm/\mathcal{M}}$; //# *reduced mass*

constants:
$K=\boldsymbol{c^2}$, $X=\sqrt{\boldsymbol{Gk_e}}$;
$U=\boldsymbol{2k_e q^2 c^3/3}$, $V=\boldsymbol{\mu c^3}$;
$W_m=\boldsymbol{G\mathcal{M}\mu c^4}$, $W_q=\boldsymbol{k_e Qqc^4}$;
$X_M=\boldsymbol{G\mathcal{M}}$, $X_m=\boldsymbol{G\mu}$;
$X_Q=X\boldsymbol{Q}$, $X_q=X\boldsymbol{q}$;
$R=X_M/K$;

initialise: $\boldsymbol{t}=0$, $\boldsymbol{r}=\boldsymbol{r_0}$, $\dot{\boldsymbol{r}}=\dot{\boldsymbol{r}}_0$, $\ddot{\boldsymbol{r}}=NaN$;
write initial values: $\{\boldsymbol{t},\boldsymbol{r},\dot{\boldsymbol{r}},\ddot{\boldsymbol{r}},R\}$; //# including stop criterion for radius
while ($\boldsymbol{r}>R\ \wedge\ \dot{\boldsymbol{r}}<0$)
{ calculate: $A=\frac{U}{\dot{r}(K-\dot{r}^2)^3}<0$;
and: $B=\frac{V}{(K-\dot{r}^2)^{3/2}}>0$;
as well as: $C=\frac{W_m}{(rK-X_M)(rK-X_m)}+\frac{W_q}{(rK-X_Q)(rK-X_q)}>0$;
//# rendering: $A\ddot{\boldsymbol{r}}^2+B\ddot{\boldsymbol{r}}+C=0$;
//# hence: $\ddot{\boldsymbol{r}}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$;
//# must choose correct sign for inward acceleration, i.e. $\ddot{\boldsymbol{r}}<0$;
since $A<0$ & $B,C>0$: $\ddot{\boldsymbol{r}}=\frac{-B+\sqrt{B^2-4AC}}{2A}$;
2nd-order Taylor step: $\Delta\boldsymbol{r}=\dot{\boldsymbol{r}}\Delta t+\ddot{\boldsymbol{r}}(\Delta t)^2/2<0$;
quit with error status if: $\neg(A<0\ \wedge\ B,C>0\ \wedge\ \ddot{\boldsymbol{r}}<0\ \wedge\ \Delta\boldsymbol{r}<0)$;
current *fluxion = velocity*: $\dot{\boldsymbol{r}}=\Delta\boldsymbol{r}/\Delta t$;
increment *radius*: $\boldsymbol{r}\mathrel{+}=\Delta\boldsymbol{r}$; *//# this actually decrements*
increment *time*: $\boldsymbol{t}\mathrel{+}=\Delta t$;
write output (at regular intervals): $\{\boldsymbol{t},\boldsymbol{r},\dot{\boldsymbol{r}},\ddot{\boldsymbol{r}}\}$;
};
write final $\{\boldsymbol{t},\boldsymbol{r},\dot{\boldsymbol{r}},\ddot{\boldsymbol{r}}\}$ & if it *has hit the thing, came to a halt or started falling upwards...*

The resulting list of $\{\boldsymbol{t},\boldsymbol{r},\dot{\boldsymbol{r}},\ddot{\boldsymbol{r}}\}$ renders $\boldsymbol{r}(\boldsymbol{t})$, $[\dot{\boldsymbol{r}}=\boldsymbol{v}](\boldsymbol{t})$, & $[\ddot{\boldsymbol{r}}=\boldsymbol{a}](\boldsymbol{t})$, $\boldsymbol{v}(\boldsymbol{r})$, etc.

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