First of all, truly incompressible fluids do not exist. They merely are an abstract concept that allows relatively simple mathematics. The latter is just what we are going to do.

The easiest *approximation* of central gravitational pressure inside a homogeneous massive ball of mass M and radius R is by splitting it into two half balls that attract one another. A *ball* is a *sphere* including its content. The center of mass of a massive hemisphere

is at[1]: $r_{\text{bc}} = 3R/8$,

so the gravitational pull is: $F = \frac{G\cdot(M/2)\cdot(M/2)}{(2r_{\text{bc}})^2} = \frac{GM^2}{4(3R/4)^2} = \frac{GM^2}{4\cdot 9R^2/16} = \frac{4GM^2}{9R^2}$

so the pressure would be: $P = \frac{F}{A} = \frac{4GM^2}{9R^2\cdot\pi R^2} = \frac{4GM^2}{9\pi R^4} = \frac{4G\left(\frac{4\pi}{3}\rho R^3\right)^2}{9\pi R^4} = \frac{4G\frac{16\pi^2}{9}\rho^2 R^6}{9\pi R^4}$

which equals: $P = \frac{64\pi G\rho^2 R^2}{81}$ [0]

where R is the full radius of the ball and P the pressure at its very centre. The above is however not very realistic. It's too quick & dirty and it does not give a result at an arbitrary depth.

Now consider a homogeneous spherical soap bubble of mass M and radius R. It obviously consists of two half bubbles & both are hemispheres. The centre of mass of a (hollow) hemisphere

is at[1]: $r_{\text{bc}} = R/2$,

so the gravitational attraction between the two hemispheres

equals: $F = \frac{G\cdot(M/2)\cdot(M/2)}{(2r_{\text{bc}})^2} = \frac{G\cdot(M/2)\cdot(M/2)}{(2R/2)^2} = \frac{GM^2}{4R^2}$.

As said, M & R are mass and radius of the bubble, which merely is a thin shell. The circumference of the great circle ("equator") where both hemispheres touch

is equal to: $L = 2\pi R$,

yielding a surface tension of[2]: $\gamma = \frac{F}{2L} = \frac{GM^2/4R^2}{4\pi R} = \frac{GM^2}{16\pi R^3}$,

where the factor of 2 is because the soap film has an inner and an outer surface. If the thickness of the soap film is neglible, it renders an internal pressure (Laplace pressure[3])

of: $P_{\text{L}} = \frac{2\gamma}{R} = \frac{GM^2}{8\pi R^4}$,

P_{L} would be the inward pressure felt by for example a homogeneous gas within the bubble, which it compensates by its own expansive pressure. Its surface area

is: $A = 4\pi R^2$,

so the corresponding force is: $F = A\cdot P_{\text{L}} = 4\pi R^2\frac{GM^2}{8\pi R^4} = \frac{GM^2}{2R^2}$,

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[1] See: https://www.vedantu.com/jee-main/physics-centre-of-mass-of-hollow-and-solid-hemisphere
[2] See: https://en.wikipedia.org/wiki/Surface_tension
[3] See: https://en.wikipedia.org/wiki/Laplace_pressure

where $M =$ the total mass of the spherical shell (i.e. soap film) and $R =$ the radius of the bubble. Would the soap film have a (very small) thickness d, and the soap's density be ρ,

then: $M = 4\pi R^2 d \cdot \rho$,

yielding: $F = \frac{G(4\pi R^2 d\cdot\rho)^2}{2R^2} = 8\pi^2 G\rho^2 (Rd)^2$.

Now we go build an "onion" of soap bubbles...

We simply take: Rd where R is the radius of a single bubble (shell)

& replace it with: $\int_D^R r dr = \frac{R^2 - D^2}{2}$,

where R is the outer radius of the onion and D is the depth, but measured as a radius.

I MUST ADMIT I HAVE NOT VERIFIED IF THIS IS THE CORRECT WAY OF DOING IT.

It would render: $F(D) = 8\pi^2 G\rho^2 \left(\frac{R^2 - D^2}{2}\right)^2 = 2\pi^2 G\rho^2 (R^2 - D^2)^2$.

Division by the surface area: $A(D) = 4\pi D^2$

yields: $P_{\text{oni}}(D) = \frac{\pi G\rho^2 (R^2 - D^2)^2}{2D^2} = \frac{\pi G\rho^2 \left(R^2\left\{1 - \frac{D^2}{R^2}\right\}\right)^2}{2D^2}$

and with: $f \coloneqq \frac{D}{R}$

we obtain the "onion" pressure: $P_{\text{oni}}(D) = \frac{\pi G\rho^2 R^4 (1 - f^2)^2}{2D^2} = \frac{\pi G\rho^2 R^2 (1 - f^2)^2}{2f^2}$,

which can be written as: $P_{\text{oni}}(D) = \frac{1}{2}\pi G\rho^2 R^2 \left(f^2 + \frac{1}{f^2} - 2\right)$. [1]

This steeply approaches infinity as f approaches nought.

We can also calculate the (potential) energy density at a given radius. In a homogeneous gravitational field, the potential energy

equals: $V_{\text{g}} = mgh$

and in a small portion of a radial field obeying Newton's law of gravitation (like here on Earth's surface) it is the same.

We then have: $g = GM/R^2$,

where: $M = \frac{4\pi}{3}\rho R^3 \therefore g = \frac{4\pi}{3} G\rho R$,

so the potential energy becomes: $V_{\text{g}} = \frac{4\pi}{3} G\rho R m h$.

Now let m be the mass of a thin shell with density ρ, and we replace R with D .

It renders: $m = \rho V_{\text{sh}}$, where V_{sh} is the shell's volume.

which in turn yields: $V_{\text{g}} = \frac{4\pi}{3} G\rho D(\rho V_{\text{sh}})h = \frac{4\pi}{3} G\rho^2 D V_{\text{sh}} h$

resulting in a local gravitational energy density, hence pressure,

of: $P_{\text{gpe},0} = \frac{V_{\text{g}}}{V_{\text{sh}}} = \frac{4\pi}{3} G\rho^2 D h$, [2]

which would apply between D and $D + h$.

It yields nought at the very centre.

It is however way better to use: $V_g = \frac{-GMm}{D}$,

which applies at the surface of an $\{M, \rho, D\}$ ball (where D is its radius). The minus sign means the resulting pressure is compressive. We'll further discard this sign.

As usual, we have: $M = \frac{4\pi}{3}\rho D^3$,

as well as: $m = \rho V_{sh}$,

rendering: $V_g = \frac{G\cdot\frac{4\pi}{3}\rho D^3\cdot\rho V_{sh}}{D} = \frac{4\pi}{3}G\rho^2 D^2 V_{sh}$.

Should we somehow integrate this to take all outer mass into account? Does the potential energy of a body change if you put another mass right on top of it? I think not (and I am not René Descartes. Oops! I think not & I am not...). The same would apply to [2].

Therefore: $P_{gpe} = \frac{V_g}{V_{sh}} = \frac{4\pi}{3}G\rho^2 D^2$. [3]

Both [2] & [3] yield *zero pressure at the very centre*, in agreement with Newton's spherical shell theorem: inside a homogenous sphere, gravitation cancels out, i.e. in a concentric spherical hollow space within a ball that is homogeneous outside it, gravitation equals nought, diddly squat, zilch (the suavity of the cavity gravity depravity). And there are more forces than gravitation to keep the fluid together. A hemispherical bubble can rest on a surface that fully carryies its weight. More or less (but maybe more less than more, since next is not a fluid) like the weight of a dome not being reveiled via its inner surface, since it fully rests on its edge The soap bubble's hemispheres feel their mutual gravitation only on their common equator, *perpendicular* to the radius.

The above results severely deviate from what is conventionally claimed to be the gravitational pressure inside a homogeneous ball of incompressible fluid. A conventional derivation via Newton's hydrostatic equilibrium with constant density

is: $dF_{shell} = \frac{GM(r)\cdot dm_{shell}}{r^2} = \frac{GM(r)\cdot\rho A_{shell}dr}{r^2}$,

so: $dP_{shell} = \frac{dF_{shell}}{A_{shell}} = \frac{GM(r)\rho dr}{r^2}$.

It is the ***pressure by*** a ***single*** infinitesimally thin ***shell***, exerted on its (touching) interior.

It renders: $\boldsymbol{\frac{dP}{dr} = \frac{GM(r)\rho}{r^2}}$, which can be found in many sources.

We find: $dP_{shell} = \frac{G\left(\frac{4\pi}{3}\rho r^3\right)\rho dr}{r^2} = \frac{4\pi}{3}G\rho^2 r dr$.

integration from surface inward: $P_{conv}(D) = \frac{4\pi}{3}G\rho^2\int_R^D r dr$,

where D is the depth, but measured as a radius, i.e. from the centre outwards.

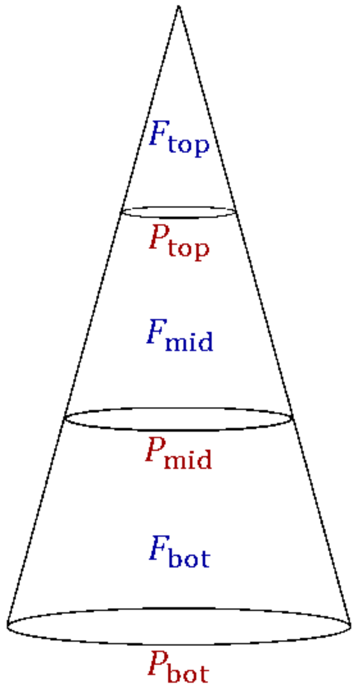
$$P_{conv}(D) = \frac{4\pi}{3}G\rho^2\cdot\frac{r^2}{2}\Big]_R^D = \frac{4\pi}{3}G\rho^2\left(\frac{D^2}{2} - \frac{R^2}{2}\right)$$

(discarding sign): $P_{conv}(D) = \frac{2\pi}{3}G\rho^2(R^2 - D^2) = \frac{2\pi}{3}G\rho^2 R^2\cdot(1 - f^2)$

which is the same as: $P_{conv}(D) = \frac{3GM^2}{8\pi R^6}(R^2 - D^2) = \frac{3GM^2}{8\pi R^4}(1 - f^2)$. [4]

As said, D grows from the centre, where ($f = 0$): $P_{conv,max} = \frac{2\pi}{3}G\rho^2 R^2 = \frac{3GM^2}{8\pi R^4}$.

But who taught you to integrate a *pressure*? Shouldn't we integrate the *force* and then convert the result to a *pressure*?



P_{xxx} should be read: $\Delta P_{xxx} =$ pressure by a single section.

$$\Delta P_{xxx} = \frac{F_{xxx}}{A_{xxx}}, \quad \sum \Delta P = \frac{F_{top}}{A_{top}} + \frac{F_{mid}}{A_{mid}} + \frac{F_{bot}}{A_{bot}}$$

$$= \frac{F_{top}A_{mid}A_{bot} + F_{mid}A_{top}A_{bot} + F_{bot}A_{top}A_{mid}}{A_{top}A_{mid}A_{bot}}$$

$$A_{top} = (1/3)^2 = 1/9, \quad A_{mid} = (2/3)^2 = 4/9, \quad A_{bot} = (3/3)^2 = 1$$

$$\therefore \sum \Delta P = \frac{\frac{4}{9}F_{top} + \frac{1}{9}F_{mid} + \frac{4}{81}F_{bot}}{4/81} = \frac{9F_{top} + \frac{9}{4}F_{mid} + F_{bot}}{1}$$

$$F_{top} = \frac{1}{3}A_{top}h\rho g = \frac{h\rho g}{27}$$ (h = height of each section)

$$F_{mid} = \frac{1}{3}A_{mid} \cdot 2h\rho g - F_{top} = \frac{7h\rho g}{27}$$

$$F_{bot} = \frac{1}{3}A_{bot} \cdot 3h\rho g - F_{top} - F_{mid} = \frac{19h\rho g}{27}$$

$$(h\rho g = 1) \Rightarrow \sum \Delta P = \frac{9}{27} + \frac{9 \cdot 7}{4 \cdot 27} + \frac{19}{27} = \frac{175}{108}$$

But: $$P = \frac{\sum F}{A_{bot}} = \frac{1}{27} + \frac{7}{27} + \frac{19}{27} = 1 \neq \frac{175}{108}$$ **Q.E.D.**

The mass up to D is: $$M(D) = \frac{4\pi}{3}\rho D^3,$$

which exerts a gravitational force on each outer shell

equal to: $$dF = G\frac{\frac{4\pi}{3}\rho D^3 . 4\pi\rho r^2 dr}{r^2} = \frac{16\pi^2}{3}G\rho^2 D^3 dr,$$

therefore: $$F(D) = \frac{16\pi^2}{3}G\rho^2 D^3 \int_R^D dr = \frac{16\pi^2}{3}G\rho^2 D^3 \cdot r]_R^D$$

$$= \frac{16\pi^2}{3}G\rho^2 D^3 (D - R),$$

rendering: $$P(D) = \frac{F(D)}{A(D)} = \frac{16\pi^2 G\rho^2 D^3 (D-R)}{3 \cdot 4\pi D^2} = \frac{4\pi G\rho^2 D(D-R)}{3},$$

(discarding sign): $$P(D) = \frac{4\pi}{3}G\rho^2 D(R - D) = \frac{4\pi}{3}G\rho^2 R^2 \frac{D}{R}\left(1 - \frac{D}{R}\right),$$

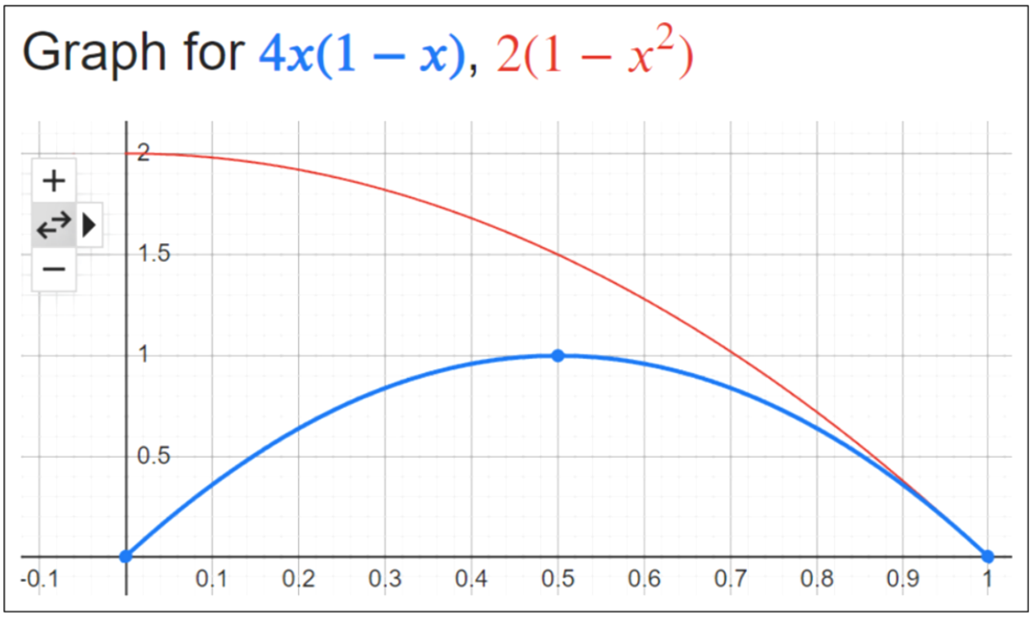
resulting in: $$P(D) = \frac{4\pi}{3}G\rho^2 R^2 \cdot f(1 - f). \quad [5]$$

Like [2] & [3], it renders nought at the very centre, in agreement with the shell theorem.

It has a max. abs. value of: $P_{max} = \frac{\pi}{3}G\rho^2 R^2 = \frac{3GM^2}{16\pi R^4} = \frac{1}{2}P_{conv,max}$ at $D = R/2$.

Please note that in [4], we have $\frac{2\pi}{3}$ vs. $\frac{4\pi}{3}$ and a factor of: $1 - f^2 = (1 + f)(1 - f)$.

Graphs of the flawed equation [4] (obtained by integrating pressure) & the correct equation [5] (obtained by integrating force) are shown below.



Newton's spherical shell theorem says:
zero gravity inside a hollow homogeneous spherical shell,
(even if the cavity has a zero radius);

I say: one should not integrate the *pressure*, but the *force*.

The blue curve is the correct one!

$$\boldsymbol{P_g = \frac{4\pi}{3} G\rho^2 R^2 \cdot f(1-f)}$$

where f is the depth as a fraction of R .

Confirming evidence by measurements on the spot
can only be obtained by drilling deeper than $R_{\oplus}/10 > 600$ km.

The deepest hole ever drilled sofar is the
Kola Superdeep Borehole at Murmansk, Russia,
but we cannot look into it. It's closed with a cap:

12226 Метров ≈ 12 km is not quite 600 km...

For *compressible* fluids, all or most of the above does not apply. The solar model described by Jørgen Christensen-Dalsgaard seems nicely confirmed by observations of — among other things — seismic solar phenomena. He obviously uses [4] to build his model[4]. Please note that the sun definitely is *not* an *in*compressible fluid and it would not shine if its central pressure (and density and temperature) were not *very* high. After all, it performs nuclear fusion, which on Earth will take at least another 30 years, as has already been the case for over 60 years...

Please also compare Earth's atmosphere. In homogeneous gravitation, P & ρ of an isothermal ideal gas incline exponentially when going down. In spherical Newtonian gravitation, a solid angle from the centre of a spherical cloud of gas outwards, has a horizontal area, hence mass over there, that increases quadratically with height, whilst gravitation decreases quadratically. It will effectively be homogeneous, thus yielding the same exponential behaviour of P & ρ. They can easily become great near the centre. Altogether, this means Dalsgaard's usage of [4] is — in spite of my claim that [4] is fundamentally wrong — in fact not so very bad an approximation.

And now, something completely different...

Electrical equivalent of Schwarzschild radius, i.e. where an electron, in free fall towards a proton, would reach the speed of light w.r.t. the proton, calculated the simple Newtonian way (which would also yield the Schwarzschild radius):

$$(V_\mathrm{e} = T) \therefore \left(k_\mathrm{e}\frac{e^2}{r^*} = \frac{1}{2}m_\mathrm{e}c^2\right)$$

$$\Rightarrow r^* = \frac{2k_\mathrm{e}e^2}{m_\mathrm{e}c^2} = 2r_\mathrm{e} \approx 2 \cdot 2.817\,940\,3262 \text{ fm},$$

which happens to be twice the classical electron radius, i.e. its classical diameter.
Dimensionless electron radius, scaled to

- proton radius: $r_\mathrm{e}/r_\mathrm{p} \approx 3.3491$;
- electron Compton wavelength: $r_\mathrm{e}/\lambda_\mathrm{C,e} \approx 1/861.022\,576\,00$;
- Bohr radius: $r_\mathrm{e}/r_\mathrm{B} \approx 1/18\,778.865\,045$.

Would this Schwarzschild look-alike be some sort of closest-possible-approach distance? Would electron capture by the proton be unavoidable if the electron comes within it? Might this be a very simple criterion for a collapse to a neutron star?

* * * * *

[4] See: J. Christensen-Dalsgaard, "Solar structure and evolution", https://arxiv.org/pdf/2007.06488.pdf ; in equations (1) & (2) @p8 one should recognise [4].

